Matrix elements of the labelling operators for SU(4) ⊃ SU(2) × SU(2)

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Matrix elements of the labelling operators for $SU(4) \supset SU(2) \times SU(2)$

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Abstract. A matrix representation of the labelling operators of Moshinsky and Nagel and of Partensky and Maguin in the non-orthogonal Draayer basis of SU(4) is derived; this allows one to solve explicitly the missing label problem for the spin-isospin multiplets in the arbitrary SU(4) supermultiplets. The simplification and degeneration of these representations are considered for two parametric irreducible representations of SU(4), as well as for some special cases of irreps typical of nuclear theory. The expansion of the linearly dependent states of the Draayer basis is discussed.

1. Introduction

The Wigner supermultiplet model continues to play an important role in nuclear spectroscopy (Gaponov *et al* 1980, 1982, Gaponov 1984). Many solutions are suggested for the missing label problems of the $SU(2) \times SU(2)$ states in the Wigner supermultiplets of SU(4). The eigenstates of the classifying operators of Moshinsky and Nagel (1963) or Partensky and Maguin (1978) may be chosen as alternatives to the non-orthogonal analytic supermultiplet bases (Draayer 1970, Ahmed and Sharp 1972, see also Norvaišas and Ališauskas 1977, Ališauskas and Norvaišas 1979, Hecht *et al* 1987). These two complete sets (couples) of commuting operators ($SU(2) \times SU(2)$ scalars) belong to the enveloping algebra of SU(4) (Quesne 1976, 1977) when the operators from the different couples do not commute. The eigenvalues of these operators have been determined only for some concrete irreducible representations (irreps) of SU(4) (Partensky and Maguin 1978) and for some non-degenerate or twofold degenerate states of the spin-isospin in SU(4) (Van der Jeugt *et al* 1983).

Also, Ališauskas and Norvaišas (1979) obtained the matrix elements of the SU(4) generators in the Draayer (1970) projected basis. Only the appearance of the linearly dependent states makes their explicit expressions more complicated.

It is the purpose of our paper to present the matrix elements of the labelling operators of Moshinsky and Nagel (1963) and Partensky and Maguin (1978) in the projected (Draayer 1970) basis for arbitrary irreps of SU(4). The solution of the eigenvalue problem together with the expression for overlaps of the projected SU(4) states (see Norvaišas 1981, Ališauskas 1982, 1983) leads to the orthonormal basis states of the supermultiplet basis.

Considerable simplifications are possible for special classes of irreps and particular cases typical of nuclear theory. In the appendix we reconsider the expansion of the linearly dependent states of the Draayer basis, which will also be necessary for calculating Clebsch-Gordan coefficients in a future publication.

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2. Definitions and notation

The infinitesimal operators (generators) of SU(4) form SU(2)×SU(2)-irreducible tensor operators with the following components:

$$\begin{aligned} &\boldsymbol{S}_0=\tfrac{1}{2}(E_{11}+E_{22}-E_{33}-E_{44}) \\ &\boldsymbol{S}_+=-(1/\sqrt{2})(E_{13}+E_{24}) \qquad \boldsymbol{S}_-=(1/\sqrt{2})(E_{31}+E_{42}) \end{aligned} \tag{2.1a}$$

$$\begin{aligned} &\boldsymbol{T}_0=\tfrac{1}{2}(E_{11}-E_{22}+E_{33}-E_{44}) \\ &\boldsymbol{T}_+=-(1/\sqrt{2})(E_{12}+E_{34}) \qquad \boldsymbol{T}_-=(1/\sqrt{2})(E_{21}+E_{43}) \end{aligned} \tag{2.1b}$$

$$\begin{aligned} &\boldsymbol{U}_{11}=E_{14} \qquad \boldsymbol{U}_{10}=(1/\sqrt{2})(E_{24}-E_{13}) \qquad \boldsymbol{U}_{1-1}=-E_{23} \\ &\boldsymbol{U}_{01}=(1/\sqrt{2})(E_{34}-E_{12}) \qquad \boldsymbol{U}_{0-1}=(1/\sqrt{2})(E_{21}-E_{43}) \\ &\boldsymbol{U}_{-11}=-E_{32} \qquad \boldsymbol{U}_{-10}=(1/\sqrt{2})(E_{31}-E_{42}) \qquad \boldsymbol{U}_{-1-1}=E_{41} \\ &\boldsymbol{U}_{00}=\tfrac{1}{2}(E_{11}-E_{22}-E_{33}+E_{44}) \end{aligned} \tag{2.1c}$$

where the E_{ij} satisfy the usual commutation relation

$$[E_{ij}, E_{kl}]=\delta_{jk}E_{il}-\delta_{il}E_{kj}. \tag{2.2}$$

Irreps of SU(4) may be denoted by triplets $(\lambda_1\lambda_2\lambda_3)$, where

$$\lambda_1=m_{14}-m_{24} \qquad \lambda_2=m_{24}-m_{34} \qquad \lambda_3=m_{34}-m_{44} \tag{2.3}$$

and $[m_{14}, m_{24}, m_{34}, m_{44}]$ is a Young tableau for an irrep of U(4). Some authors use the parameters $[pp'p'']$ of irreps of the locally isomorphic group SO(6) where

$$p=\tfrac{1}{2}(\lambda_1+\lambda_3)+\lambda_2 \qquad p'=\tfrac{1}{2}(\lambda_1+\lambda_3) \qquad p''=\tfrac{1}{2}(\lambda_1-\lambda_3). \tag{2.4}$$

The states of the non-orthonormal Draayer (1970) basis are defined as

$$\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E \\ K_SSM_S;\, K_TTM_T\end{matrix}\right\rangle=\boldsymbol{P}^S_{M_SK_S}\boldsymbol{P}^T_{M_TK_T}|G_E\{K_SK_T\}\rangle \tag{2.5}$$

where $\boldsymbol{P}^S_{M_SK_S}$, $\boldsymbol{P}^T_{M_TK_T}$ are the projection operators of the two SU(2) and

$$|G_E\{K_SK_T\}\rangle=\left|\begin{matrix}\lambda_1+\lambda_2+\lambda_3 & & \lambda_2+\lambda_3 & & \lambda_3 & & 0 \\ & k_1+\tfrac{1}{2}\lambda_1+\lambda_2+\lambda_3 & & \lambda_3 & & 0 & \\ & & k_1+\tfrac{1}{2}\lambda_1+\lambda_2+\lambda_3 & & \tfrac{1}{2}\lambda_3+k_3 & & \\ & & & k_1+\tfrac{1}{2}\lambda_1+\lambda_2+\lambda_3 & & & \end{matrix}\right\rangle \tag{2.6}$$

are special Gel'fand-Zetlin states, which form the basis of irrep $(\tfrac{1}{2}\lambda_1, \tfrac{1}{2}\lambda_3)$ of the block-diagonal subgroup SU(2)⊕SU(2) generated by $\boldsymbol{U}_{11}$, $\boldsymbol{U}_{-11}$, $\boldsymbol{U}_{1-1}$, $\boldsymbol{U}_{-1-1}$ and their commutators. Here $k_1=\tfrac{1}{2}(K_S+K_T)$, $k_3=\tfrac{1}{2}(K_S-K_T)$, and

$$S\leqslant p \qquad T\leqslant p \qquad S+T\leqslant p+p' \qquad |S-T|\leqslant p\pm p'' \tag{2.7}$$

and $\tfrac{1}{2}\lambda_1\pm k_1$, $\tfrac{1}{2}\lambda_3\pm k_3$ are non-negative integers.

The linearly independent states (2.5) are determined by conditions (cf Draayer 1970) $k_1\geqslant 0$ ($k_3\leqslant 0$ if $k_1=0$) and, when $S-T>\lambda_2$,

$$K_S\geqslant S-\lambda_2 \tag{2.8a}$$

or, when $T-S>\lambda_2$,

$$K_T\geqslant T-\lambda_2 \tag{2.8b}$$

and (separately for each sign of k_3)

$$k_1 + |k_3| \geq \tfrac{1}{2}[S + T - \lambda_2 + 1 + (1 - \Delta_0)\,\mathrm{sgn}(k_3 - \tfrac{1}{4})] \tag{2.8c}$$

when $|S - T| \leq \lambda_2$. Here $\Delta_0 = 1$ or 1 and $\lambda_2 - S + T - \Delta_0$ is an even integer. The values $K_S = K_T = 0$ are allowed only when $S + T - \lambda_2 \leq 0$ is an even integer. Some linearly dependent states may be eliminated by means of the relation (3.12) of Ališauskas and Norvaišas (1979):

$$\left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ -K_S S M_S;\ -K_T T M_T \end{matrix} \right\rangle = (-1)^{\lambda_3 + \Delta_0} \left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ K_S S M_S;\ K_T T M_T \end{matrix} \right\rangle. \tag{2.9}$$

The labelling operators (scalars with respect to the spin and isospin) of Moshinsky and Nagel (1963) may be defined as

$$\boldsymbol{\Omega} = 3[\boldsymbol{S} \times [\boldsymbol{T} \times \boldsymbol{U}]^{1,0}]^{0,0} \tag{2.10a}$$

$$\boldsymbol{\Phi} = \boldsymbol{s} + \boldsymbol{t} + 2\boldsymbol{S}^2\boldsymbol{T}^2 + 6[\boldsymbol{S} \times \boldsymbol{T} \times [\boldsymbol{U} \times \boldsymbol{U}]^{1,1}]^{0,0} \tag{2.10b}$$

where $\boldsymbol{S}^2$, $\boldsymbol{T}^2$ are the Casimir operators of the two SU(2) and

$$\boldsymbol{s} = -3\sqrt{3}[[\boldsymbol{T} \times \boldsymbol{U}]^{1,0} \times [\boldsymbol{T} \times \boldsymbol{U}]^{1,0}]^{0,0} \tag{2.10c}$$

$$\boldsymbol{t} = -3\sqrt{3}[[\boldsymbol{S} \times \boldsymbol{U}]^{0,1} \times [\boldsymbol{S} \times \boldsymbol{U}]^{0,1}]^{0,0} \tag{2.10d}$$

are the labelling operators of Partensky and Maguin (1978). The coupling of the double SU(2) tensors $\boldsymbol{W}^{k,l}$ is defined according to Jucys and Bandzaitis (1977):

$$[\boldsymbol{W}^{k,l} \times \boldsymbol{W}^{k',l'}]^{k''l''}_{m''n''} = \sum_{mm'nn'} \begin{bmatrix} k & k' & k'' \\ m & m' & m'' \end{bmatrix} \begin{bmatrix} l & l' & l'' \\ n & n' & n'' \end{bmatrix} \boldsymbol{W}^{k,l}_{m,n} \boldsymbol{W}^{k',l'}_{m',n'} \tag{2.11}$$

where, in the RHS, the Clebsch-Gordan coefficients of SU(2) are used.

3. Representation of the labelling operators

Ališauskas and Norvaišas (1979) obtained the following representation of SU(4) generators $\boldsymbol{U}_{i,j}$ as the Draayer basis:

$$\begin{aligned} \boldsymbol{U}_{ij} \left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ K_S S M_S;\ K_T T M_T \end{matrix} \right] &= \sum_{S'T'} \frac{(2S+1)(2T+1)}{(2S'+1)(2T'+1)} \begin{bmatrix} S & 1 & S' \\ M_S & i & M'_S \end{bmatrix} \\ &\times \begin{bmatrix} T & 1 & T' \\ M_T & j & M'_T \end{bmatrix} \sum_{mnK'_SK'_T} \mathscr{C}^{(\lambda_1\lambda_2\lambda_3)}_{SK_STK_T;\,S'K'_ST'K'_T} \begin{bmatrix} S & 1 & S' \\ K_S & m & K'_S \end{bmatrix} \\ &\times \begin{bmatrix} T & 1 & T' \\ K_T & n & K'_T \end{bmatrix} \left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ K'_S S' M'_S;\ K'_T T' M'_T \end{matrix} \right\rangle \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \mathscr{C}^{(\lambda_1\lambda_2\lambda_3)}_{SK_STK_T;\,S'K'_ST'K'_T} &= (-1)^{k_3 - k'_3}(1 - \delta_{K_SK'_S}\delta_{K_TK'_T}) H(\tfrac{1}{2}\lambda_1, k_1k'_1) H(\tfrac{1}{2}\lambda_3, k_3k'_3) \\ &\quad + \delta_{K_SK'_S}\delta_{K_TK'_T}[S'(S'+1) + T'(T'+1) - S(S+1) - T(T+1) + 2p + 4] \end{aligned} \tag{3.2}$$

and

$$H(j, mm') = \delta_{mm'} + \delta_{m\pm1,m'}[(j \mp m)(j \pm m + 1)]^{1/2}. \tag{3.3}$$

Equation (3.1) has been derived with the help of the commutation relation between the tensor and projective operators (cf Elliott 1958). The operators U_{10}, U_{-10}, U_{01} and U_{0-1} may be replaced by $-\sqrt{2}S_+$, $\sqrt{2}S_-$, $-\sqrt{2}T_+$ and $\sqrt{2}T_-$, respectively, when acting on special Gel'fand-Zetlin states (2.6). Later they may be included in the projection operators. The remaining operators U_{00}, U_{11}, U_{-1-1}, U_{1-1}, and U_{-11} belong to the Lie algebra of the block-diagonal subgroup U(2)⊕U(2), the basis of the irrep $(\frac{1}{2}\lambda_1, \frac{1}{2}\lambda_3)$ of which is formed by special Gel'fand-Zetlin states (2.6).

The scalar operators commute with the projection operator. Therefore a rather long, though elementary, evaluation allows one to represent the action of the scalars included in the labelling operators as follows:

$$\begin{aligned}\Omega\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K_SSM_S;\,K_TTM_T\end{matrix}\right\rangle &= \sum_{K'_SK'_T}[K_SK_T(p+2)\delta_{K_SK'_S}\delta_{K_TK'_T}\\ &\quad + H(\tfrac{1}{2}\lambda_1, k_1k'_1)H(\tfrac{1}{2}\lambda_3, k_3k'_3)H(S, K_SK'_S)H(T, K_TK'_T)\\ &\quad \times\tfrac{1}{2}(1-\delta_{K_SK'_S}\delta_{K_TK'_T})]\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K'_SSM_S;\,K'_TTM_T\end{matrix}\right\rangle\end{aligned} \tag{3.4}$$

$$\begin{aligned}(\Phi-s-t)&\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K_SSM_S;\,K_TTM_T\end{matrix}\right\rangle\\ &= \sum_{K'_SK'_T}\{2[p''(p'+1)K_SK_T + K_S^2T(T+1) + K_T^2S(S+1) - 2K_S^2K_T^2]\\ &\quad\times\delta_{K_SK'_S}\delta_{K_TK'_T} + (-1)^{k_3-k'_3}(1-\delta_{K_SK'_S}\delta_{K_TK'_T})\\ &\quad\times H(\tfrac{1}{2}\lambda_1, k_1k'_1)H(\tfrac{1}{2}\lambda_3, k_3k'_3)H(S, K_SK'_S)H(T, K_TK'_T)\\ &\quad\times[p+1+K_S(K_S-K'_S)+K_T(K_T-K'_T)]\}\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K'_SSM_S;\,K'_TTM_T\end{matrix}\right\rangle\end{aligned} \tag{3.5}$$

$$\begin{aligned}s\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K_SSM_S;\,K_TTM_T\end{matrix}\right\rangle &= \sum_{K'_SK'_T}\Big(\{[T(T+1)-K_T^2][\tfrac{1}{4}\lambda_1(\lambda_1+2)+\tfrac{1}{4}\lambda_3(\lambda_3+2)-\tfrac{1}{2}(K_S^2+K_T^2)]\\ &\quad + K_T^2[(p+2)^2 - S(S+1) + K_S^2]\}\delta_{K_SK'_S}\delta_{K_TK'_T}\\ &\quad -(1-\delta_{K_SK'_S}\delta_{K_TK'_T})[\tfrac{2}{3}(2T-1)(2T+3)T(T+1)]^{1/2}\\ &\quad\times 2^{-|K'_S-K_S|}H(\tfrac{1}{2}\lambda_1, k_1k'_1)H(\tfrac{1}{2}\lambda_3, k_3k'_3)H(S;\,K_SK'_S)\\ &\quad\times(-1)^{k_3-k'_3}\begin{bmatrix}T & 2 & T\\ K_T & K'_T-K_T & K'_T\end{bmatrix}\Big)\left|\begin{matrix}(\lambda_1\lambda_2\lambda_3)_E\\ K'_SSM_S;\,K'_TTM_T\end{matrix}\right\rangle.\end{aligned} \tag{3.6}$$

The action of the operator t may be written after the formal substitution $S \leftrightarrow T$, $K_S \leftrightarrow K_T$, $K'_S \leftrightarrow K'_T$ into (3.6) together with (2.9), if necessary. (The dependence of the non-diagonal matrix elements (3.5) on K_T, K'_T is included in the Clebsch-Gordan coefficients of SU(2).) The labelling operators may change both parameters K_S, K_T by ±1 or a single parameter by ±2. Therefore, the linearly dependent states which appear should be expanded in terms of the complete system with the help of the results of § 5 of Ališauskas and Norvaišas (1979)† and the relation (2.9) is insufficient (also see the appendix).

4. Simplification of the labelling operators for two parametric representations

The analytical bases for irreps of class 2 (covariant) and $1,\bar{1}$ (mixed tensor) (i.e. when $\lambda_3=0$ or $\lambda_2=0$) have been considered by Ališauskas (1982, 1983, 1984, 1987) and Petrauskas and Ališauskas (1987). Thus, for the class 2 irreps of SU(4), the Draayer basis is labelled by a single parameter $K=K_S=K_T\geqslant\frac{1}{2}(S+T-\lambda_2+\Delta_0)$. The operator $\mathbf{\Omega}$ in this case is represented by a quasitridiagonal matrix because in (3.4) the matrix elements with $K'_S\neq K'_T$ vanish and the linearly dependent states may be expanded by means of (5.1) and (5.6*a*) of Ališauskas and Norvaišas (1979) or our equation (2.9). The operator $\mathbf{\Phi}$ in the basis of the irreps with $\lambda_3=0$ is dependent:

$$\mathbf{\Phi}=2(p+2)\mathbf{\Omega}+\tfrac{1}{4}\lambda_1(\lambda_1+2)[S(S+1)+T(T+1)]\mathbf{I} \tag{4.1}$$

where $\mathbf{I}$ is the identity operator.

The operator $\boldsymbol{s}$ for the irreps of class 2 may be represented as

$$\begin{aligned}\boldsymbol{s}\left|\begin{matrix}(\lambda_1\lambda_2 0)_E\\ KSM_S;\,KTM_T\end{matrix}\right\rangle &=\{[T(T+1)-K^2][\tfrac{1}{4}\lambda_1(\lambda_2+2)-K^2]+K^2[(p+2)^2-S(S+1)+K^2]\}\\ &\quad\times\left|\begin{matrix}(\lambda_1\lambda_2 0)_E\\ KSM_S;\,KTM_T\end{matrix}\right\rangle+\tfrac{1}{2}\sum_{K'}(1-\delta_{KK'})(-1)^{(K-K'+1)/2}\\ &\quad\times(K+K')H(\tfrac{1}{2}\lambda_1,KK')H(S,KK')H(T,KK')\\ &\quad\times\left|\begin{matrix}(\lambda_1\lambda_2 0)_E\\ K'SM_S;\,K'TM_T\end{matrix}\right\rangle\end{aligned} \tag{4.2}$$

and

$$\boldsymbol{t}-\boldsymbol{s}=\tfrac{1}{4}\lambda_1(\lambda_1+2)[S(S+1)-T(T+1)]\mathbf{I}. \tag{4.3}$$

The matrix elements of the labelling operators may similarly be simplified for $\lambda_1=0$.

The linearly independent states of the class $1,\bar{1}$ irreps may be determined by condition $K_2=S$ (if $S\geqslant T$), but the expansion of the dependent states is more complicated. The projected states in this case are equivalent (up to norm) to the stretched states, which are dual to the polynomial states (see Ališauskas 1984, 1987). Therefore, it is expedient to write the tridiagonal representation of the labelling operators in the complete (non-overcomplete) basis

$$\begin{aligned}\mathbf{\Omega}\left|\begin{matrix}(\lambda_1 0\lambda_3)_E\\ SSM_S;\,K_TTM_T\end{matrix}\right\rangle &=\sum_{K'_T}\Big[\{p''T(T+1)+\tfrac{1}{2}K_T[S(2p+3)-T(T+1)+p+2]\\ &\quad-\tfrac{1}{2}p''K_T^2+\tfrac{1}{2}K_T^3\}\delta_{K_TK'_T}-\delta_{K_T-2,K'_T}H(\tfrac{1}{2}\lambda_1,k_1k'_1)\\ &\quad\times\left(\frac{\frac{1}{2}\lambda_3-k_3}{\frac{1}{2}\lambda_3+k_3+1}\right)^{1/2}\tilde{H}(T,K_T)+\delta_{K_T+2,K'_T}H(\tfrac{1}{2}\lambda_3,k_3k'_3)\\ &\quad\times\left(\frac{\frac{1}{2}\lambda_1-k_1}{\frac{1}{2}\lambda_1+k_1+1}\right)^{1/2}\tilde{H}(T,-K_T)\Big]\left|\begin{matrix}(\lambda_1 0\lambda_3)_E\\ SSM_S;\,K'_TTM_T\end{matrix}\right\rangle\end{aligned} \tag{4.4}$$

† The variable $\underline{k}_3$ in the RHS of equation (5.4*a*) of Ališauskas and Norvaišas (1979), and $|k_3|$ in the RHS of (5.5*c*) should both be corrected to $|\underline{k}_3|$. In the RHS of (5.6*a*), $k_1^{\Delta_0}$ should be corrected to $\underline{k}_1^{\Delta_0}$.

$$
\begin{aligned}
&{}_S\left|\begin{matrix}(\lambda_1 0\lambda_3)_E \\ SSM_S;\, K_T TM_T\end{matrix}\right\rangle \\
&\quad = \sum_{K'_T} \Big[\{\tfrac{1}{4}T(T+1)[(\lambda_1+1)(\lambda_1+2)+(\lambda_3+1)(\lambda_3+2)-2S(S+1)] \\
&\quad - p''K_T[T(T+1)-\tfrac{1}{2}]+\tfrac{1}{2}K_T^2[\lambda_1\lambda_3+3p+1+S(S+1) \\
&\quad + T(T+1)]+p''K_T^3-\tfrac{1}{2}K_T^4\}\delta_{K_T K'_T} \\
&\quad + \delta_{K_T-2,K'_T}\tfrac{1}{2}(\lambda_3+S+K_T+1)\left(\frac{\frac{1}{2}\lambda_3-k_3}{\frac{1}{2}\lambda_3+k_3+1}\right)^{1/2} H(\tfrac{1}{2}\lambda_1, k_1k'_1)\tilde{H}(T, K_T) \\
&\quad + \delta_{K_T+2,K'}\tfrac{1}{2}(\lambda_1+S-K_T+1)\left(\frac{\frac{1}{2}\lambda_1-k_1}{\frac{1}{2}\lambda_1+k_1+1}\right)^{1/2} H(\tfrac{1}{2}\lambda_1, k_3k'_3)\tilde{H}(T, -K_T)\Big] \\
&\quad \times \left|\begin{matrix}(\lambda_1 0\lambda_3)_E \\ SSM_S;\, K'_T TM_T\end{matrix}\right\rangle
\end{aligned}
\tag{4.5}
$$

where

$$\tilde{H}(j, m) = [(j+m)(j+m-1)(j-m+1)(j-m+2)]^{1/2}. \tag{4.6}$$

The operators $\mathbf{\Phi}$ and $\boldsymbol{t}$ in the case of the irreps $(\lambda_1 0\lambda_3)$ are dependent as well:

$$\mathbf{\Phi} = 2p''\mathbf{\Omega}+(p+1)(p+2)[S(S+1)-T(T+1)]\mathbf{I} \tag{4.7}$$

$$\boldsymbol{t}-\boldsymbol{s} = (p+1)(p+2)[S(S+1)-T(T+1)]\mathbf{I}. \tag{4.8}$$

5. Some examples

Let us present the expansion coefficients of the $SU(4)\supset SU(2)\times SU(2)$ irreducible tensors in terms of basis states (2.5) as columns. Then it will be convenient to represent the labelling operators $\mathcal{O}$ as matrices $\mathcal{O}^{K'_S K'_T}{}_{K_S K_T}$ with subscripts for rows and superscripts for columns.

In the situation typical from the physical point of view, the parameters λ_1 and λ_3 are small while λ_2, S and T are arbitrary. For example, when $\lambda_1 = 2$ and $\lambda_3 = 0$ we obtain the matrix elements

$$
\begin{aligned}
&\Omega^{11}{}_{11} = p+2 \qquad \Omega^{00}{}_{00} = 0 \qquad \Omega^{00}{}_{11} = [\tfrac{1}{2}S(S+1)T(T+1)]^{1/2} \\
&\Omega^{11}{}_{00} = [1+(-1)^{\Delta_0}]\Omega^{00}{}_{11}
\end{aligned}
\tag{5.1}
$$

$$
\begin{aligned}
&S^{11}{}_{11} = T(T+1)-S(S+1)+(p+2)^2 \qquad S^{00}{}_{00} = 2T(T+1) \\
&S^{00}{}_{11} = -\Omega^{00}{}_{11} \qquad S^{11}{}_{00} = \Omega^{11}{}_{00}.
\end{aligned}
\tag{5.2}
$$

and the eigenvalues (for $\Delta_0 = 0$)

$$\Omega_\pm = \tfrac{1}{2}(p+2)\pm[\tfrac{1}{4}(p+2)^2+S(S+1)T(T+1)]^{1/2} \tag{5.3}$$

$$
\begin{aligned}
S_\pm = \tfrac{1}{2}(S^{00}{}_{00}+S^{11}{}_{11})\pm\{\tfrac{1}{4}[(p+2)^2-S(S+1)-T(T+1)]^2 \\
-S(S+1)T(T+1)\}^{1/2}.
\end{aligned}
\tag{5.4}
$$

(The value $K=0$ is impossible for $\Delta_0 = 1$. A single value $K=1$ remains for $S+T=\lambda_2+2$ and $K=0$ remains for $S=0$ or $T=0$.)

When $\lambda_1 = \lambda_3 = 1$ we obtain the matrix elements

$$\Omega^{10}{}_{01} = \Omega^{01}{}_{10} = [1-(-1)^{\Delta_0}]\tfrac{1}{2}[S(S+1)T(T+1)]^{1/2} \qquad \Omega^{10}{}_{10} = \Omega^{01}{}_{01} = 0 \tag{5.5}$$

$$\Phi^{10}{}_{01} = \Phi^{01}{}_{10} = -[1+(-1)^{\Delta_0}](p+2)[S(S+1)T(T+1)]^{1/2}$$

$$\Phi^{10}{}_{10} = 2T(T+1) + 2\Delta_0 S(S+1) + (p+2)^2 - 1 \tag{5.6}$$

$$\Phi^{01}{}_{01} = 2S(S+1) + 2\Delta_0 T(T+1) + (p+2)^2 - 1$$

$$S^{01}{}_{01} = T(T+1)[1-(-1)^{\Delta_0}] - S(S+1) + (p+2)^2 - 1 \qquad S^{10}{}_{10} = T(T+1)$$

$$S^{10}{}_{01} = -S^{01}{}_{10} = [1+(-1)^{\Delta_0}]\tfrac{1}{2}[S(S+1)T(T+1)]^{1/2}. \tag{5.7}$$

In this case the classifying properties of the labelling operators depend on the parity of Δ_0. For $\Delta_0 = 0$ the operator $\mathbf{\Omega}$ is degenerate but the eigenvalues of $\mathbf{\Phi}$, i.e.

$$\Phi_{\pm} = S(S+1) + T(T+1) + (p+2)^2 - 1$$
$$\pm([S(S+1) - T(T+1)]^2 + 4(p+2)^2 S(S+1)T(T+1))^{1/2} \tag{5.8}$$

are different when operators $\boldsymbol{s}$ and $\boldsymbol{t}$ $(s + t = (p+2)^2 - 1)$ have the same discriminant:

$$\tfrac{1}{4}[(p+2)^2 - 1 - S(S+1) - T(T+1)]^2 - S(S+1)T(T+1) \tag{5.9}$$

of the eigenvalue problem. For $\Delta_0 = 1$ operator $\mathbf{\Phi}$ is degenerate when the eigenvalues of $\mathbf{\Omega}$, i.e.

$$\Omega_{\pm} = \pm[S(S+1)T(T+1)]^{1/2} \tag{5.10}$$

are different; the operators $\boldsymbol{s}$ and $\boldsymbol{t}$ are represented in this case by diagonal matrices, i.e. the Draayer basis is orthogonal.

Let us also write the matrix elements of the labelling operators in the case $\lambda_1 = 2$, $\lambda_3 = 1$. The matrices of $\mathbf{\Omega}$ and $\mathbf{\Phi}$ in this case are symmetric:

$$\Omega^{\frac32\frac12}{}_{\frac32\frac12} = \Omega^{\frac12\frac32}{}_{\frac12\frac32} = \tfrac34(p+2)$$

$$\Omega^{\frac12-\frac12}{}_{\frac12-\frac12} = -\tfrac14(p+2) - (-1)^{\Delta_0}\tfrac12(S+\tfrac12)(T+\tfrac12)$$

$$\Omega^{\frac12\frac32}{}_{\frac32\frac12} = \Omega^{\frac32\frac12}{}_{\frac12\frac32} = \tfrac12[(S+\tfrac32)(S-\tfrac12)(T+\tfrac32)(T-\tfrac12)]^{1/2} \tag{5.11}$$

$$\Omega^{\frac12-\frac12}{}_{\frac32\frac12} = \Omega^{\frac32\frac12}{}_{\frac12-\frac12} = (T+\tfrac12)[\tfrac12(S+\tfrac32)(S-\tfrac12)]^{1/2}$$

$$\Omega^{\frac12-\frac12}{}_{\frac12\frac32} = \Omega^{\frac12\frac32}{}_{\frac12-\frac12} = (-1)^{\Delta_0+1}(S+\tfrac12)[\tfrac12(T+\tfrac32)(T-\tfrac12)]^{1/2}$$

$$\Phi^{\frac32\frac12}{}_{\frac32\frac12} = \tfrac14[15T(T+1) + 7S(S+1) + 10(p+2)^2 - 12]$$

$$\Phi^{\frac12\frac32}{}_{\frac12\frac32} = \tfrac14[7T(T+1) + 15S(S+1) + 10(p+2)^2 - 12]$$

$$\Phi^{\frac12-\frac12}{}_{\frac12-\frac12} = \tfrac14[11T(T+1) + 11S(S+1) + 2(p+2)^2 - 8] + (-1)^{\Delta_0}(p+2)(S+\tfrac12)(T+\tfrac12)$$

$$\Phi^{\frac12\frac32}{}_{\frac32\frac12} = \Phi^{\frac32\frac12}{}_{\frac12\frac32} = -(p+2)[(S+\tfrac32)(S-\tfrac12)(T+\tfrac32)(T-\tfrac12)]^{1/2} \tag{5.12}$$

$$\Phi^{\frac12-\frac12}{}_{\frac32\frac12} = [(p+2)(T+\tfrac12) - (-1)^{\Delta_0}(S+\tfrac12)][2(S+\tfrac32)(S-\tfrac12)]^{1/2}$$

$$\Phi^{\frac12-\frac12}{}_{\frac12\frac32} = [T+\tfrac12 - (-1)^{\Delta_0}(p+2)(S+\tfrac12)][2(T+\tfrac32)(T-\tfrac12)]^{1/2}.$$

For operator $\boldsymbol{s}$ we obtain

$$S^{\frac32\frac12}{}_{\frac32\frac12} = \tfrac14[6T(T+1) - S(S+1) + (p+2)^2 + \tfrac34]$$

$$S^{\frac12\frac32}{}_{\frac12\frac32} = \tfrac34[2T(T+1) - 3S(S+1) + 3(p+2)^2 - \tfrac{15}{4}]$$

$$S^{\frac12-\frac12}{}_{\frac12-\frac12} = \tfrac14[10T(T+1) - S(S+1) + (p+2)^2 - \tfrac94]$$

$$S^{\frac32\frac12}{}_{\frac12\frac32} = -S^{\frac12\frac32}{}_{\frac32\frac12} = -[(S+\tfrac32)(S-\tfrac12)(T+\tfrac32)(T-\tfrac12)]^{1/2} \tag{5.13}$$

$$S^{\frac12\frac32}{}_{\frac12-\frac12} = [T+\tfrac12 - (-1)^{\Delta_0}(S+\tfrac12)][2(T+\tfrac32)(T-\tfrac12)]^{1/2}$$

$$S^{\frac12-\frac12}{}_{\frac12\frac32} = [T+\tfrac12 + (-1)^{\Delta_0}(S+\tfrac12)][2(T+\tfrac32)(T-\tfrac12)]^{1/2}$$

$$S^{\frac32\frac12}{}_{\frac12-\frac12} = S^{\frac12-\frac12}{}_{\frac32\frac12} = 0.$$

The matrices with the elements (5.11) and (5.12) commute mutually as well as those of the operators *s* and *t*, but the eigenvalues of the operators are non-degenerate. The formal solutions of the eigenvalue problems in this case are rather cumbersome.

The terms above with factor $(-1)^{\Delta_0}$ appeared after applying (2.9). Sometimes (e.g. when $S+T=\lambda_1+\lambda_2+\lambda_3$) the labels of the linearly dependent states appear between superscripts in (5.1), (5.2), (5.5)-(5.7), (5.11)-(5.14). In this case the zero eigenstates (i.e. the null spaces) correspond to certain eigenvalues.

Of course, in the multiplicity-free cases the corresponding discriminants in (5.3), (5.4), (5.8) and (5.9) become exact squares. Since for the fixed couple of parameters from the set λ_1, λ_3, S, T the multiplicity of S, T in $(\lambda_1\lambda_2\lambda_3)$ is restricted, equation (2.9) together with the results of § 3 is sufficient for the eigenvalue problem of the labelling operators.

The expansion of the linearly dependent states is unavoidable when the restriction of the multiplicity of S, T in $(\lambda_1\lambda_2\lambda_3)$ is caused by the fixed value of any single parameter between the following linear combinations:

$$p+p'-S-T \qquad p-T \qquad p-S \tag{5.14a}$$

$$p-|p''|-|T-S|. \tag{5.14b}$$

Particularly, for $p+p'=S+T$ or $p-T=0$ the labels K_S, K_T accept the values

$$K_S=p'' \qquad K_T=p'. \tag{5.15a}$$

For $p+p'-1=S+T$ they accept (5.15*a*) together with

$$K_S=p' \qquad K_T=p''. \tag{5.15b}$$

For $p-T=1$ and $|p''|+1\leqslant S<p'-1$ they accept (5.15*a*) together with

$$K_S=p''\pm 1 \qquad K_T=p'-1. \tag{5.15c}$$

For $p-p''=T-S$ $(p''\geqslant 0)$ they accept

$$K_S=S \qquad K_T=T-\lambda_2=S+\lambda_2 \tag{5.15d}$$

and for $p-p''-1=T-S$ $(p''\geqslant 0)$ they take on the values

$$K_S=S,\ K_T=T-\lambda_2+1 \qquad \text{and} \qquad K_S=S-1,\ K_T=T-\lambda_2. \tag{5.15e}$$

As an example of the second kind we present the matrix elements of the labelling operators for $S+T=p+p'-1$:

$$\begin{aligned}
&\Omega^{p'p''}{}_{p'p''}-\Omega^{p''p'}{}_{p''p'}=p''(S-T)\\
&\Omega^{p'p''}{}_{p''p'}=(-1)^{\lambda_3+1}(S+1)(T-p'+1)Z\\
&\Omega^{p''p'}{}_{p'p''}=(-1)^{\lambda_3+1}(T+1)(S-p'+1)Z^{-1}
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
&S^{p'p''}{}_{p'p''}-S^{p''p'}{}_{p''p'}=(2T+3)[p''^2-p'(p+1)]\\
&S^{p'p''}{}_{p''p'}=(-1)^{\lambda_3+1}(2T+3)p''(T-p'+1)Z\\
&S^{p''p'}{}_{p'p''}=(-1)^{\lambda_3}(2T+3)p''(S-p'+1)Z^{-1}
\end{aligned} \tag{5.17}$$

where

$$Z=\left(\frac{(S-p'')!(S+p'')!(T-p')!(T+p')!}{(S-p')!(S+p')!(T-p'')!(T+p'')!}\right)^{1/2}$$

together with corresponding eigenvalues

$$\Omega_{\pm} = p''[p'(p+2)+(S-p'+1)(T-p'+1)+\tfrac{1}{2}(\lambda_2+1)] \pm [\tfrac{1}{4}p''^2(S-T)^2+(S+1)(T+1)(S-p'+1)(T-p'+1)]^{1/2} \tag{5.18}$$

$$S_{\pm} = p''^2[(p+2)^2+S(S+1)+p'(p'-2S-1)-(p-T)(2p+3)] + [p(p'-1)+T](T+1)+\tfrac{1}{2}(2T+3)[p'(p+1)-p''^2] \pm \tfrac{1}{2}(2T+3)([p'(p+1)-p''^2]^2 - 4p''^2(S-p'+1)(T-p'+1))^{1/2}. \tag{5.19}$$

The operators $\mathbf{\Phi}$ and $\mathbf{\Omega}$ are dependent in this case:

$$\mathbf{\Phi}+2p''\mathbf{\Omega} = [S(S+1)+T(T+1)](p''^2+1)-(p+p')(p+p'+1) + p''^2[2p'(p+2)+2(S-p'+1)(T-p'+1)+3p+p'+4] + 2(S+1)(T+1)(pp'-p-1)+p'(p+1)(3p+3p'+4). \tag{5.20}$$

For $S+T=p+p'-1$ and $\lambda_1=\lambda_3$ the Draayer states are the eigenstates of the operators $\boldsymbol{s}$ and $\boldsymbol{t}$ but the operator $\mathbf{\Phi}$ is degenerate. In the above we omitted the matrix elements of $\boldsymbol{t}$ that are related to those of $\boldsymbol{s}$ by the corresponding substitutions (see § 3).

6. Concluding remarks

We obtained rather simple explicit expressions for the matrix elements of the labelling operators in the Draayer basis of the SU(4) ⊃ SU(2) × SU(2) chain. For the normalisation of the eigenstates and the construction of the coupling coefficients the overlaps of the Draayer states are necessary. They will be considered in a future publication. However, for multiplicities larger than two, only a numerical solution of the eigenvalue problems of the labelling operators is possible. For some classes of irreps we demonstrated that the definite labelling operators become degenerate, but we have not found a general rule governing these relations.

It is difficult to estimate the advantages of the diagonalisation of the overlap matrix (see Hecht *et al* 1987) in comparison with the solution of the eigenvalue problem of the labelling operators.

Appendix. On the expansion of the linearly dependent states of the Draayer basis

The linearly dependent states (2.6) may be expanded in two steps, introducing an auxiliary basis (Ahmed and Sharp 1972, Ališauskas and Norvaišas 1979). Let us write the special generalised isofactors of SU(4) ⊃ SU(2) × SU(2):

$$\sum_{\omega} \begin{bmatrix} (\lambda_1 00) & (00\lambda_3) & (\lambda_1 0\lambda_3) \\ s_1 s_1 & s_3 s_3 & \omega,\, s_0 t_0 \end{bmatrix} \begin{bmatrix} (\lambda_1 0\lambda_3) & (0\lambda_2 0) & (\lambda_1\lambda_2\lambda_3)_E \\ \omega,\, s_0 t_0 & s_2 t_2 & K_S S;\, K_T T \end{bmatrix} = (-1)^{s_3+k_3} C^{\lambda_1}_{k_1 s_1} C^{\lambda_3}_{k_3 s_3} \tilde{C}^{\lambda_2}_{s_2 t_2} \begin{bmatrix} s_1 & s_3 & s_0 \\ k_1 & k_3 & K_S \end{bmatrix} \times \begin{bmatrix} s_1 & s_3 & t_0 \\ k_1 & -k_3 & K_T \end{bmatrix} \begin{bmatrix} s_0 & s_2 & S \\ K_S & 0 & K_S \end{bmatrix} \begin{bmatrix} t_0 & t_2 & T \\ K_T & 0 & K_T \end{bmatrix} \tag{A1}$$

which couple the orthonormal states of the symmetric irreps $(\lambda_1 00)$, $(00\lambda_3)$ and $(0\lambda_2 0)$ to non-orthonormal Draayer states. Here

$$C^{\lambda}_{ks}=\left[\frac{(\frac{1}{2}\lambda+k)!(\frac{1}{2}\lambda-k)!(2s+1)}{(\frac{1}{2}\lambda-s)!(\frac{1}{2}\lambda+s+1)!}\right]^{1/2} \tag{A2a}$$

$$\tilde{C}^{\lambda}_{st}=\left[\frac{2(2s+1)(2t+1)\lambda!(\lambda+1)!}{(\lambda-s-t)!!(\lambda-s+t+1)!!(\lambda+s-t+1)!!(\lambda+s+t+2)!!}\right]^{1/2}. \tag{A2b}$$

We introduce the direct and inverse expansion matrices **A** and **Ω**. The expansion coefficients $A^{(\lambda_1\lambda_2\lambda_3;ST)}_{k_1k_3;s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2)}$ of the Draayer states in terms of the auxiliary basis states may be obtained by substituting into (A1) the following values of the parameters. For $S>T$, $s_1+s_3>T$ we substitute

$$s_2=S+\Delta-s_1-s_3 \qquad t_2=0 \qquad t_0=T \qquad s_0=s_1+s_3. \tag{A3a}$$

For $S<T$, $s_1+s_3>S$ we substitute

$$s_2=0 \qquad t_2=T+\bar{\Delta}-s_1-s_3 \qquad s_0=S \qquad t_0=s_1+s_3. \tag{A3b}$$

For $s_1+s_3\leqslant\min(S,T)$ we substitute

$$s_2=S-s_1-s_3+1-\delta_{\Delta_0\Delta_2} \qquad t_2=T-t_0 \qquad s_0=t_0+\Delta_2=s_1+s_3 \tag{A3c}$$

unless $\lambda_1-\lambda_3=0 \bmod 2$, $\Delta_0=1$. For $s_1+s_3\leqslant\min(S,T)$, $\Delta_0=1$ we substitute

$$\begin{gathered} s_0=s_1+s_3-1+\tilde{\Delta}_2 \qquad t_0=s_1+s_3-\tilde{\Delta}_2 \qquad s_2=S-s_1-s_3 \\ t_2=T-s_1-s_3 \qquad s_1>0 \qquad s_3>0. \end{gathered} \tag{A3d}$$

For $0<s_3\leqslant\min(S,T)$, $\Delta_0=1$, $s_1=0$ we substitute

$$s_2=S-s_3+1 \qquad t_2=T-t_0 \qquad s_0=t_0=s_3 \tag{A3e}$$

and for $0<s_1\leqslant\min(S,T)$, $\Delta_0=1$, $s_3=0$ we substitute

$$s_2=S-s_1+1 \qquad t_2=T-t_0 \qquad s_0=t_0=s_1. \tag{A3f}$$

The last two conditions, (A3*e*) and (A3*f*), represent particular cases of (A3*c*). Here and below Δ, $\bar{\Delta}$, Δ_0, Δ_2 and $\tilde{\Delta}_2$ are equal to 0 or 1 so that

$$s_1+s_3-S+\lambda_2-\Delta \qquad s_1+s_3-T+\lambda_2-\bar{\Delta} \qquad \lambda_2-S+T-\Delta_0 \tag{A4}$$

are even integers.

Now the parameters s_1, s_3 and (if necessary) Δ_2 or $\tilde{\Delta}_2$ may be used as the labels of an auxiliary basis† and the coefficients $A^{(\lambda_1\lambda_2\lambda_3;ST)}_{k_1k_3;s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2)}$ form a triangular expansion matrix. The elements of this matrix with $k_1>s_1$ or $|k_3|>s_3$ or $k_1=s_1$, $k_3=-s_3$, $\Delta_2=1$ ($\tilde{\Delta}_2=1$) vanish. The parameters are also restricted by the condition (cf (2.8))

$$s_1+s_3\geqslant\max(S,T)-\lambda_2 \tag{A5a}$$

$$s_1+s_3\geqslant\tfrac{1}{2}[S+T-\lambda_2+1+(1-\Delta_0)\Delta_2]. \tag{A5b}$$

† It is possible to choose from two alternatives (A3*c*) or (A3*d*) for $\lambda_1-\lambda_3\neq 0 \bmod 2$, $\Delta_0=1$.

We succeeded in inverting this matrix explicitly† and finding the following expansion coefficients, Q, of the auxiliary basis in the different regions of parameters:

$$Q'^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3;k_1k_3} = n'^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3(\bar\Delta)}(-1)^{k_1-s_1}(C^{\lambda_1}_{k_1s_1}C^{\lambda_3}_{k_3s_3})^{-1}$$
$$\times\left(\frac{(S+K_S)!(S-K_S)!}{(T+K_T)!(T-K_T)!}\right)^{-1/2}\frac{(s_1+s_3+S+1)}{(s_1+s_3+1-\bar\Delta)}$$
$$\times\sum_y\frac{(s_1+s_3+S+y-1)!}{y!(k_1-s_1-y)!(-k_3-s_3-y)!(S+s_3-k_1+y)!(S+s_1+k_3+y)!}$$
$$\times[(s_1+s_3+S)(2s_1+2s_3-k_1+k_3+1)^{1-\bar\Delta}$$
$$+2(1-\bar\Delta)y(2s_1+2s_3+1)] \qquad k_3\leqslant 0,\ T\geqslant K_T>s \tag{A6}$$

$$Q''^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3\Delta_2;k_1k_3} = n''^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3\Delta_2(\Delta_0)}(-1)^{k_3+s_3}(C^{\lambda_1}_{k_1s_1}C^{\lambda_3}_{k_3s_3})^{-1}$$
$$\times[(S+K_S)!(S-K_S)!(T+K_T)!(T-K_T)!]^{-1/2}\Bigg((1-\delta_{k_10})(1-\delta_{k_30})$$
$$\times\frac{2\,\mathrm{sgn}(k_3-\frac14)(-1)^{k_1-s_1+|k_3|-s_3}(k_1+s_1-1)!(|k_3|+s_3-1)!}{[s_1+(-1)^{\Delta_0}s_3+(1-\Delta_0)(1-\Delta_2)](k_1-s_1)!(|k_3|-s_3)!}$$
$$\times[(k_1-k_3)^{\delta}(s_1k_3-s_3k_1)^{1-\Delta_2}+(k_1k_3-s_1s_3)(1-\Delta_0)(1-\Delta_2)]$$
$$+\delta_{k_30}\delta_{s_30}(1-\delta_{k_10})\delta_{\Delta_20}(-1)^{k_1-s_1}2k_1^{1-\Delta_0}(k_1+s_1-1)!/(k_1-s_1)!$$
$$-\delta_{k_10}\delta_{s_10}(1-\delta_{k_30})\delta_{\Delta_20}(-1)^{k_3+s_3}2k_3^{1-\Delta_0}(-k_3+s_3-1)!/(-k_3-s_3)!$$
$$+\delta_{k_10}\delta_{s_10}\delta_{k_30}\delta_{s_30}\delta_{\Delta_20}(1-\Delta_0)\Bigg) \qquad \delta=\delta_{\Delta_0\Delta_2},\ k_1+|k_3|\leqslant\min(S,T) \tag{A7}$$

or

$$\bar Q''^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3\tilde\Delta_2;k_1k_3} = \bar Q''^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3,1-\tilde\Delta_2;k_1,-k_3}$$
$$=\bar n''^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3\tilde\Delta_2}[(S+K_S)!(S-K_S)!(T+K_T)!(T-K_T)!]^{-1/2}$$
$$\times(-1)^{\lambda_3+k_1-s_1}(C^{\lambda_1}_{k_1s_1}C^{\lambda_3}_{k_3s_3})^{-1}\frac{(k_1+s_1-1)!(k_3+s_3-1)!}{s_1s_3(k_1-s_1)!(k_3-s_3)!}$$
$$\times[s_1k_3-(-1)^{\tilde\Delta_2}k_1s_3]$$
$$k_3>0,\ k_1+|k_3|\leqslant\min(S,T),\ \Delta_0=1,\ s_1>0,\ s_3>0. \tag{A8}$$

The substitution of

$$S\leftrightarrow T \qquad K_S\leftrightarrow K_T \qquad k_3\to -k_3 \qquad \bar\Delta\to\Delta \tag{A9}$$

into (A6) (together with the additional factor $(-1)^{\lambda_3}$) allows us to obtain $Q^{(\lambda_1\lambda_2\lambda_3;ST)}_{s_1s_3;k_1k_3}$ also for $S\geqslant K_S>T$, $k_3\geqslant 0$.

The normalisation factors n', n'', $\bar n''$ in (A6)-(A8) may be written such as to take into account the fact that the diagonal matrix elements of $\mathbf{A}$ and $\mathbf{Q}$ (with $s_1=k_1$, $s_3=|k_3|$, $\mathrm{sgn}(k_3-\frac14)=2\Delta_2-1$ or $2\tilde\Delta_2-1$) are mutually inverse.

† The advantage of our choice of the auxiliary parameters to compare with Ahmed and Sharp (1972) is in the appearance of the stretched or almost stretched Clebsch-Gordan coefficients of SU(2) in (A1) corresponding to the (A3c) or (A3d) cases.

Generally, the auxiliary labels may accept values in several regions (A3*a*)-(A3*e*). (Only (A3*a*) and (A3*b*) are mutually exclusive, as well as (A3*c*) and (A3*d*).) When the labels k_1k_3 and s_1s_3 are taken from different regions the inverse expansion coefficients take on more complete forms, for example:

$$\tilde{Q}''^{(\lambda_1\lambda_2\lambda_3;ST)}_{0s_3;k_1k_3} = \sum_{k_3'\tilde{\Delta}_2's_1's_3'} Q''_{0s_3;0-k_3'} \bar{A}''_{0,-k_3';s_1's_3'\tilde{\Delta}_2'} \bar{Q}''_{s_1's_3'\tilde{\Delta}_2';k_1k_3} \qquad \text{(A10)}$$

where $s_1' > 0$, $k_1 > 0$, $k_1 + |k_3| \leqslant \min(S, T)$ (see also (3.11) of Ališauskas and Norvaišas 1979). Luckily, they are not necessary for our main purpose, namely the expansion of the linearly dependent states.

The expansion coefficients $R^{(\lambda_1\lambda_2\lambda_3;ST)}_{\underline{k}_1\underline{k}_3;k_1k_3}$ of the linearly dependent states (labelled by $\underline{K}_S\underline{K}_T$ or $\underline{k}_1\underline{k}_3$) may be found after multiplication of the matrices $A_{\underline{k}_1\underline{k}_3;s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2)}$ and $Q_{s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2);k_1k_3}$, where the summation parameters s_1, s_3 (and Δ_2 or $\tilde{\Delta}_2$) are restricted by the condition (A5*a*) or (A5*b*). Since these conditions are concealed as the factorial factors in n', n'' or $\bar{n}''$, the sum continued to the whole region is equal to zero. Thus we obtain

$$R^{(\lambda_1\lambda_2\lambda_3;ST)}_{\underline{k}_1\underline{k}_3;k_1k_3} = -\sum_{s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2)} A_{\underline{k}_1\underline{k}_3;s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2)} Q_{s_1s_3(\Delta_2 \text{ or } \tilde{\Delta}_2);k_1k_3} \qquad \text{(A11)}$$

where for $|T-S| > \lambda_2$ we use the condition $s_1 + s_3 < \max(S, T) - \lambda_2$ together with the coefficients $A'_{\underline{k}_1\underline{k}_3;s_1s_3}$ expressed according to (A1) with (A2*b*) or (A2*a*) and $Q'_{s_1s_3;k_1k_3}$ expressed according to (A6) (and (A9)), and for $|T-S| \leqslant \lambda_2 + 1$ we use the condition $s_1 + s_3 < \frac{1}{2}[S + T - \lambda_2 + 1 + (1-\Delta_0)\Delta_2]$ together with $\overset{(-)}{A}{}''_{\underline{k}_1\underline{k}_3;s_1s_3\Delta_2(\tilde{\Delta}_2)}$ expressed according to (A1) with (A2*c*) or (A2*d*) and $\overset{(-)}{Q}{}''_{s_1s_3\Delta_2(\tilde{\Delta}_2);k_1k_3}$ expressed according to (A7) or (A8). In all cases the factors n', n'' or n'' in **A** and **Q** should be concealed. In some regions (for $|T-S| = \lambda_2 + 1$ or $S - T = \lambda_2 + 2$), where the different solutions join, the preference should be given to the last versions.

We should emphasise that the explicit expressions of some $R_{\underline{k}_1\underline{k}_3;k_1k_3}$ are available (Ališauskas and Norvaišas 1979) for the parameters $\underline{K}_S$ or $\underline{K}_T$ at the distance 1 or 2 units from the region of the linearly independent state labels as well as for $\underline{k}_1 = k_1 = 0$ or $\underline{k}_3 = k_3 = 0$. Equations (5.1), (5.2), (5.4), (5.5) and (5.7) of Ališauskas and Norvaišas (1979) may also be used for the recursive expansion of the linearly dependent states, because the same coefficients R appear in the more general expansion

$$\left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ \underline{K}_SSM_S;\ \underline{K}_TTM_T \end{matrix} \right\rangle = \sum_{K_SK_T} R^{(\lambda_1\lambda_2'\lambda_3;ST)}_{\underline{k}_1\underline{k}_3;k_1k_3} \left| \begin{matrix} (\lambda_1\lambda_2\lambda_3)_E \\ K_SSM_S;\ K_TTM_T \end{matrix} \right\rangle \qquad \text{(A12)}$$

where $\lambda_2' \geqslant \lambda_2$ ($\lambda_2' - \lambda_2$ is even for $|T-S| \leqslant \lambda_2' + 1$) and for $\lambda_2' > \lambda_2$ some linearly dependent states are also included in the RHS. These linearly dependent states may be expanded in subsequent steps by means of the substitution of the same equations.

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